Engineering Notes

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A Consistent Approach for Treating Distributed Loading in the Matrix Force Method

L. W. REHFIELD*
Georgia Institute of Technology, Atlanta, Ga.

Introduction

ANALYSIS of complex structures by the matrix force method and the matrix displacement method is routine procedure in the aerospace industry. Both approaches are on firm conceptual ground and operational computer programs are readily available. Although there appears to be a move toward use of the displacement method in recent years, the force method remains popular with some engineering teams as it usually leads to the use of fewer unknowns. Hence larger structural systems can be handled, provided a systematic method of selecting redundants is used. An early account of the force method may be found in Ref. 1, while more recent descriptions are given in Refs. 2 and 3.

If distributed loads or loads applied at other than node points are present, there is a method for defining equivalent loads at the nodes based upon virtual work equivalence in the matrix displacement method.[4] There appears to be no corresponding consistent approach to the treatment of such loads in the matrix force method. For example, the book by Robinson[3] is devoted primarily to the force method, and distributed loads are treated by requiring static equivalence.

In this Note a viewpoint is adopted which permits distributed loads to be handled in a manner that is consistent with energy principles. The effects of distributed loads or loads applied at other than node points are shown to be equivalent to initial strains for which analytical techniques have been developed.

Outline of the Approach

For simplicity of presentation the method will be developed for a simple beam element subjected to a distributed load $q(x)$. The element is pictured in Fig. 1. M_1, M_2, S_1 and S_2 are terminal bending moments and shear forces, respectively. Insofar as it is convenient, the notation of Ref. 1 will be adopted.

In the absence of the distributed load $q(x)$ the bending moment and shear force distributions as functions of the coordinate x are

$$[M(x)]_{q=0} = M_1 + (M_2 - M_1)x/l \tag{1}$$

$$S(x) = S_1 = S_1 = (M_2 - M_1)/l \tag{2}$$

where l is the beam element length. If in addition to the terminal forces and moments the loading q is present, we can write

$$M(x) = \tilde{M}_q(x) + [M(x)]_{q=0} \tag{3}$$

The bending moment $\tilde{M}_q(x)$ is sufficient to maintain equilibrium of the beam element with $q(x)$ applied and with M_1 and M_2 zero. In physical terms it is the moment that would

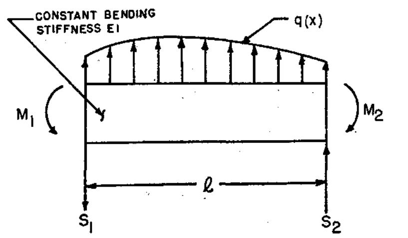

Fig. 1 Beam element acted upon by a distributed load.

exist if the beam element were simply supported at the ends and may be deduced by statics alone.

The complementary strain energy of elastic deformation (numerically equal to the strain energy) for a uniform, Hookean elastic beam element may be written

$$U^* = (1/2EI)\int_0^{\ell} (\tilde{M}_q(x) + [M(x)]_{q=0})^2\, dx \tag{4}$$

The effects of shearing deformations have been neglected. The generalized displacements that correspond to the "generalized stresses" M_1 and M_2 are the terminal rotations, β_1 and β_2. These rotations may be calculated from the following equations

$$\beta_1 = \frac{\partial U^*}{\partial M_1} = \frac{1}{EI}\int_0^{\ell} (\tilde{M}_q(x) + [M(x)]_{q=0})\left(1 - \frac{x}{\ell}\right) dx \tag{5}$$

$$\beta_2 = \frac{\partial U^*}{\partial M_2} = \frac{1}{EI}\int_0^{\ell} (M_q(x) + [M(x)]_{q=0})\frac{x}{\ell}\, dx \tag{6}$$

The preceding may be written in matrix form in the following way:

$$\underset{2\times 1}{e} \equiv \begin{Bmatrix}\beta_1\\ \beta_2\end{Bmatrix} = \underset{2\times 1}{h} + \underset{2\times 2}{f}\;\underset{2\times 1}{s} \tag{7}$$

where f is the ordinary flexibility matrix

$$f = \frac{\ell}{3EI}\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix} \tag{8}$$

s is the generalized stress matrix given by

$$\underset{2\times 1}{s} = \begin{Bmatrix}M_1\\ M_2\end{Bmatrix} \tag{9}$$

and

$$\underset{2\times 1}{h} = \frac{1}{EI}\begin{Bmatrix}\int_0^{\ell} \tilde{M}_q(x)\left(1 - \frac{x}{\ell}\right) dx\\ \int_0^{\ell} \tilde{M}_q(x)\frac{x}{\ell}\, dx\end{Bmatrix} \tag{10}$$

e, of course, is the matrix of generalized strains for the element.

The effects of the distributed loading $q(x)$ are contained in h and consequently, they are conceptually equivalent to initial strains. The usual problems of frameworks subjected to distributed loads can now be consistently formulated and solved.

Conclusions

An approach to the treatment of distributed loadings or loads applied at other than node points has been developed for the matrix force method. Although presented within an

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* Associate Professor of Aerospace Engineering. Member AIAA.

elementary context provided by a simple beam element, it may be readily applied to other, more complex elements. All that is required is to imagine the element supported in a statically determinant manner that corresponds to $s = 0$, determine the internal stress distribution in this configuration under the loads and evaluate the complementary strain energy. The effects will have the appearance of initial element strains. It is believed that this approach fills a theoretical need for an energy-consistent treatment of such loads in the matrix force method.

References

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[3] Robinson, J. S., *Structural Matrix Analysis for the Engineer*, Wiley, New York, 1966.

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Aerodynamic Characteristics of the Slotted Fin

P. DANIELS* AND T. A. CLARE†
Naval Weapons Laboratory, Dahlgren, Va.

Nomenclature

- A_F = planform area of slotted fin
- A^* = planform area of solid fin
- $C1$ = induced rolling moment coefficient $L(\phi)/qsd$
- $C1_4$ = amplitude of induced rolling moment coefficient
- d = missile body diam
- $L(\phi)$ = roll torque due to roll angle
- $L(\dot{\phi})$ = roll torque due to rolling velocity
- P = spin rate
- q = dynamic pressure
- S = $\pi d^2/4$
- V = velocity
- δ = fin cant angle

Introduction

THE flight performance of finned bodies is critically dependent on the configurations roll behavior. Because of manufacturing tolerances, slight configurational asymmetries dictate the need for spin to avoid large dispersion. Too low a design spin rate, however, may lead to resonance oscillations,[1] where the trim angle of attack due to asymmetry is amplified to a value inversely proportional to the total damping in the system. Finned configurations near resonance have been observed to develop extremely large angular motions in excess of that predicted by resonance theory alone. This "castastrophic yaw," arising from roll-induced aerodynamic moments causing "roll lock-in" or "lunar motion," was first described by Schneller[2] and later documented during the flight trials of low drag bomb configurations[3]. Magnus instability,[1] generally characterized by large rolling velocities, was noted even earlier by Kent of the Ballistic Research Laboratories.

In 1961, Lugt[4] indicated that slots, or gaps, in the fin planform might radically alter the dynamic angular motion of finned bodies by sweeping away strong wake vortices ordinarily attached to the receding fin at very large angles of attack. Pursuing this possibility, it was shown how the performance of such a basic configuration in free rolling motion responds to fin slots at all angles of attack; it was suggested that these results could be used to alleviate the problem of catastrophic yaw for finned configurations in free flight.[5,6]

These wind-tunnel tests were conducted primarily to demonstrate the feasibility of the slotted fin. Further testing has been performed on a larger sample of slotted fin configurations. It is the purpose of this paper to present a summary of these test results and discuss the effect of slot size on both roll behavior and longitudinal stability.

Wind-tunnel tests

Subsonic wind-tunnel tests were conducted at the Naval Academy to determine the effect of slot size on the longitudinal stability and rolling characteristics of a cruciform finned missile. The tests were conducted at approximately 150 FPS. The test specimen is shown in Fig. 1.

The Naval Academy model had a 3.2 caliber ogive nose with a 4.4 caliber cylindrical afterbody. The model's maximum body diam. was 1.5 in. The fins were rectangular and trapezoidal, with an exposed semispan of 1 caliber.

A free rolling test was conducted to determine the effect of fin slots and fin cant on roll lock-in and roll speed-up.

Figure 2 gives the steady-state spin rate vs angle of attack for the Naval Academy model with solid rectangular fins and approximately no fin cant.

Lock-in exists from 20° to 50° angle of attack. Considerable speed-up exists above 50°. Dual modes of motion exist throughout. Very slow clockwise motion existed below 20° and was not recorded. The fin cant was then varied from zero to a maximum of 8° in order to overcome the lock-in.

Figure 3 gives the steady-state spin rate vs angle of attack for the Naval Academy model with solid rectangular fins and 8° of fin cant. Lock-in now occurs at 30° rather than at 20° which is generally beneficial. The speed-up is hardly affected. Not only is the spin high at high angles of attack but it is also independent of the fin cant.

These results are typical for all solid fins which have been tested. The motion of the Naval Academy model was then investigated with varying slot size and fin cant.

The addition of the slot eliminates roll speed-up if the slot is sufficiently large. Figure 4 gives the motion for the minimum slot size which alleviated roll speed-up for the rectangular fin configuration. Fin cant is nearly zero.

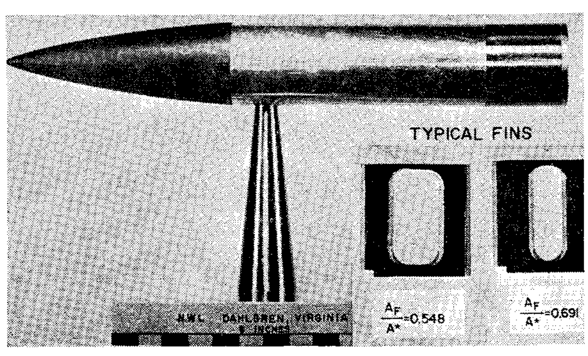

Fig. 1 Wind-tunnel model-Naval Academy.

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* Research Scientist, Warfare Analysis Department.
† Research Scientist, Warfare Analysis Department. Member AIAA.